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RESOURCES FOR TEACHING DISCRETE MATHEMATICS

Classroom Projects, History Modules, and Articles

Brian Hopkins, **EDITOR**

Mathematical Association of America

Resources for Teaching Discrete Mathematics

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Introduction

For some twenty years now, the MAA Notes Series has published secondary materials for undergraduate mathematics courses, such as projects that can be used in teaching calculus. These publications reflect the interests of instructors, providing a means of sharing innovative ideas for teaching calculus, linear algebra, differential equations, statistics, geometry, and abstract algebra. With this book, discrete mathematics joins the list. This collection includes nineteen classroom-tested projects, eleven additional projects based on historical sources, three expository articles considering discrete mathematics topics in more depth, and two articles focused on pedagogy especially related to discrete mathematics.

Why is discrete mathematics only now the subject of such a collection? One possible reason is that, unlike concepts taught in the course itself, discrete mathematics is not well-defined. While there are controversies on how to teach calculus, there is relative unanimity on what topics a calculus course should address. On the other hand, a survey of discrete mathematics courses around the country shows a variety of different topics being covered, goals being sought, and students being served. Does the course cover circuit design and the tools for algorithm analysis? How much logic is covered? Graph theory? Combinatorics? Is this the course where students first learn to write proofs? Are the students mathematics majors, computer science majors, or is the course offered for a general education requirement?

This book does not address those questions. The projects and articles here reflect the wide breadth of topics taught in the diverse discrete mathematics courses offered in universities, colleges, and (increasingly) high schools. I hope that every instructor of discrete mathematics will find projects and articles relevant to the topics of his or her course, and also learn more about other topics, with the realization that those are covered in someone else's course.

The timing of this collection also follows from a number of related activities in the professional organizations. There have recently been workshops and many sessions devoted to discrete mathematics in conferences of the Mathematical Association of America, the Association for Computing Machinery, and the Institute of Electrical and Electronics Engineers. I participated in several of these workshops and sessions, initially with the idea of writing a textbook. But as I discovered the innovative and thoughtful work of so many colleagues, I decided to redirect my creative impulse into putting together this collection. The call went out for classroom-tested projects and articles addressing advanced discrete mathematics topics and teaching issues related to the course. I received submissions from faculty in mathematics departments and computer science departments, from a high school teacher, from new instructors to experts in the field. I am grateful that a wide variety of instructors were willing to provide their work for this volume.

Most of the responses were classroom-tested projects, which vary widely in difficulty and are sometimes distinguished from the advanced articles only by the inclusion of exercises (and solutions!). Some are means of introducing a topic, such as graph theory, strong induction, and motivation for clearly written proofs. Some extend common topics, such as the Towers of Hanoi, the Josephus problem, and Euler's formula. Some discuss applications, such as chemistry, bioinformatics, information storage, and typesetting. Some use technology, including graphing calculators and programming. Some use manipulatives, including integer rods, strings on a pegboard, and pipes from the hardware store. In addition to exercises and solutions in all of these projects, some include open-ended questions and some extension questions suitable for student research. The format for each is a summary, notes to the instructor, references, student worksheets (often mixed with explanatory handouts), and solutions. The classroom projects are ordered topically, following the order of Susanna Epp's *Discrete Mathematics with Applications*.

I am pleased to be able to include eleven of the historical modules developed through New Mexico State University, a continuation of their long-standing program of teaching mathematics from original sources. These too vary from introductory to advanced topics, including combinatorics, set theory, logic, and graph theory. The projects are arranged chronologically by their primary source, allowing readers to follow the historical development of certain discrete mathematics topics. The initial two related projects, though, do not fit into that structure, going from Leibniz to von Neumann in the first, and from Shannon to the abacus in the second. A combined introduction explores how these projects can be used in the classroom, and each article includes exercises in conjunction with the source material, references, and additional notes to the instructor.

The five expository articles examine discrete mathematics content beyond the level of a first course or discuss teaching issues specific to the course. But like many of the projects, some of the articles also defy easy categorization. For example, Shai Simonson's "A Rabbi, Three Sums, and Three Problems" uses fourteenth century mathematics as a springboard for counting problems as an example of the discovery method. To assist the reader in navigating such rich material, the table of contents is annotated with a summary of each project or article and, for the classroom projects, how many 50-minute periods an instructor might dedicate to the activity.

There are many people I want to thank. Bill Marion organized two summer Professional Enhancement Programs on discrete mathematics, where I met Peter Henderson and Susanna Epp; all three have been very helpful and encouraging over the long course of this book's creation. Jerry Lodder, another PREP participant, coordinated work on the historical projects. Michael Jones and Larry Thomas helped with sundry typesetting and graphics issues. Barbara Reynolds, Stephen Maurer, the entire MAA Notes editorial board, and the MAA publications staff have helped immensely in bringing this project to completion. My greatest appreciation goes to the authors of the projects and articles for their patience, creativity, and willingness to share their good work.

In memory of Kenneth P. Bogart, 1943–2005

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Part I

Classroom-tested Projects

The Game of "Take Away"

Mark MacLean
Seattle University

Summary

In this project, students play the game "Take Away" and conceive a winning strategy for the game. They then must give a careful written explanation of why their strategies work. I typically use this project in my discrete math class as an introduction to writing proofs. Students often struggle initially with the clarity of their mathematical exposition and with aiming their proofs at an appropriate audience. After completing this activity, we have a class discussion where we critique the students' written explanations.

In most discrete math classes, students' first proofs involve properties of even and odd integers. Since these properties are already familiar to them, students often have trouble discerning what they can and cannot assume that the intended audience knows. Since the game "Take Away" is unfamiliar to most students, I find this project is a more natural starting point from which they begin thinking about the clarity of their own writing.

Alternately (or concurrently), this project could be used as an application of the division algorithm.

Notes for the instructor

Begin class by dividing the students into groups of three or four, and then pass out the worksheet. Allow one 50-minute class period to complete the worksheet, and have the students finish it at home if necessary.

Be sure to walk around the room and observe the groups as they are playing the game. If a group becomes overly frustrated, I might suggest that they try starting with 6 tokens instead of 9, or I suggest that they challenge another group to a game. I usually try to ensure every team knows the winning strategy within 25 minutes. At the end of class I instruct them to continue working at home and to bring a polished written explanation to problem #2 to the next class, as we will critique them. I encourage the students to actually show their explanations to their roommates to see if their writing is clear enough for the roommate to follow the logic.

The next class period, I ask three or four volunteers to write their explanations to question 2 on the board. Over the next 20 minutes, I ask the class to critique these explanations and point out instances where the exposition is unclear, while making some comments myself.

"Take Away" is a variant of the combinatorial game Nim. More information is available in [1].

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[1] Berlekamp, Elwyn R., John H. Conway, and Richard K. Guy. *Winning Ways for Your Mathematical Plays*, volume 1, A K Peters, Ltd., 2001.

Worksheet for The Game of "Take Away"

Rules of the game: To play the game, your group will need a pile of nine "tokens." These tokens can be coins, paper clips, small pieces of paper, or any small objects of your choosing. Divide your group into two teams. The two teams will take turns removing tokens from the pile. On each turn, a team must choose to remove either one or two tokens from the pile. The team that removes the last token (or tokens) from the pile wins the game.

Play several rounds of the game, keeping in mind the first problem below.

1. Find a strategy that guarantees your team will win the game. (Note: the existence of such a strategy will depend on whether your team goes first or second. Could there be a guaranteed winning strategy for **both** the first and second team?)
2. In a full paragraph, describe your winning strategy and carefully explain why it works. Pretend that you are explaining the strategy to your roommate, who we'll say knows the rules of the game but has only played a couple of times. Your paragraph must clearly spell out the strategy and convince the roommate that it works.
3. Can you devise a winning strategy if the game starts with 10 tokens? 11? 12? How about n tokens, where n is any positive integer? Describe these winning strategies.
4. In the television series "Survivor: Thailand," the two tribes participated in a competition entitled "Thai 21." In this contest, 21 flags were placed in a circle, and the two tribes took turns removing either 1, 2, or 3 flags from the circle. The team who removed the last flag (or flags) won the game. Is there a winning strategy for Thai 21? Explain.

Solutions

1. There is a winning strategy for the team that goes second. Hence there is no winning strategy for the team that goes first.

2. On each turn, the second team should remove the "opposite" of what the first team removed. That is, if the first team removes 1 token from the pile, the second team should then remove 2 tokens. If the first team removes 2 tokens from the pile, the second team should remove 1. Thus, after each pair of turns, a total of 3 tokens has been removed from the pile, leaving the first team to always choose when the number of tokens in the pile is a multiple of 3. Initially the first team chooses when 9 tokens are in the pile, then they choose when there are 6 tokens, and finally they choose when there are 3 tokens left. In their final turn, the second team takes the remaining tokens and wins the game.

3. If n is congruent to 0 modulo 3: choose to go second, and on each turn always take the opposite of what your opponents just took. If n is congruent to 1 modulo 3: choose to go first, take 1 token, and on each successive turn, always take the opposite of what your opponents just took. If n is congruent to 2 modulo 3: go first, take 2 tokens, then always take the opposite of your opponents. In each strategy you are always leaving your opponent to choose from a multiple of 3. (If students are not familiar with the "mod" terminology, you may simply say that n is equal to 1 plus some multiple of 3, etc.)

4. Here, you want to force your opponent to choose from a multiple of 4. The team that goes first can guarantee themselves a win if they remove 1 flag from the circle. Then, on each successive turn, if their opponents removed m flags, the first team should remove $4 - m$ flags.

Pile Splitting Problem: Introducing Strong Induction

Bill Marion
Valparaiso University

Summary

In many textbooks in discrete mathematics there are numerous examples for teaching the Weak Form of the Principle of Mathematical Induction, but relatively few elementary problems for applying the Strong Form. What follows is a nice example to draw on when introducing the strong form. It can be presented as a classic puzzle, it has a number of variants and it is inherently recursive.

By introducing the problem (Pile Splitting) as a puzzle, the instructor can engage the students in the process of finding a general solution. She can, then, raise the question as to how they can demonstrate that their conjecture is correct, and, thereby, motivate the need for strong induction. After an induction argument has been presented (the Worksheet includes one such proof), variants of the puzzle can be assigned for the students to work on in class or as a homework assignment.

Notes for the instructor

To give students practice in making conjectures about the solution to the puzzle, they should be asked to solve it themselves. One hands-on approach that works well is to provide each student with a sufficient number of beads or pennies for her to actively play the game enough times with different values of n so as to see a pattern emerge. Those students who correctly conjecture the general solution can assist the others. One way of reaching the conjecture is explained on the Solutions page. During the induction phase a template or "script" for a correct induction argument should be presented to the students (and they should be required to follow the script) so that they become more confident in using the technique and more convinced logically that induction does what we claim it does. The Worksheet includes a complete proof for one puzzle variant in order to give students another example.

In addition to the four variants of the puzzle described in this paper, there are a number of others which can be found in [2]. It should be noted that the author of that article presents a different approach which does not explicitly make use of mathematical induction to prove the general solutions correct.

The recursive nature of the pile splitting problem can lead to a discussion of recursive definitions, recurrence relations, techniques for solving recurrence relations and constructing recursive algorithms to compute the solutions. In the latter case, strong induction comes into play again. It addresses the question: "Does such and such a recursive algorithm, which is designed to compute something, actually do so?"

Bibliography

[1] Rosen, Kenneth. *Discrete Mathematics and Its Applications* 5th ed., McGraw Hill, 2003.

[2] Tanton, James. "A Dozen Questions About: Pile Splitting," *Math Horizons* 12 (2004) 28–31.

Worksheet on the Pile Splitting Problem

Here is a statement of a common version of the pile splitting problem.

> Given n objects in a pile, split the objects into two smaller piles. Continue to split each pile into two smaller piles until there are n piles of size one. At each splitting, compute the product of the size of the two smaller piles. Once there are n piles, sum all the products computed. The result will always be the same no matter how each of the piles is split into two smaller piles. The sum of products is a function of n. Conjecture what the sum is and prove the conjecture correct. [1]

The solution to this pile splitting problem is $(n^2 - n)/2$. Playing with a few examples can provide the necessary insight to come up with the general solution. Applying the strong form of the principle of mathematical induction can demonstrate the correctness of the conjecture, and, *equally as important, that the computation will always produce the same result no matter how the piles are split*.

For the following variant on the standard pile splitting problem, a complete proof of the general solution is provided below.

> We begin with a pile of n objects and proceed to reduce the pile to n piles of size one in the manner described above. Suppose at each splitting the sizes of the two smaller piles are labeled r and s. Now, instead of just computing the product $(r \cdot s)$ of the size of each pair of split piles, we compute the following product: $(r \cdot s) \cdot (r + s)$. And at the end of the process we add all these products. Again, the sum of products turns out to be a function of n: $(n^3 - n)/3$.

Here is the induction argument.

Let $P(n)$ be: for a pile consisting of n stones and split according to the rules above, the sum of all products of the sum and product of each pair of split piles is $(n^3 - n)/3$.

Show $(\forall n \geq 1)\ P(n)$.

Basis Step Show $P(n)$ is true when $n = 1$.

When $n = 1$, there are no splits; hence the product of sums and products is 0. For the formula, when $n = 1$,

$$\frac{n^3 - n}{3} = \frac{1^3 - 1}{3} = 0.$$

Inductive Step Suppose for any $k > 1$ that $P(1), P(2), \ldots, P(k-1)$ are true. Show this implies that $P(k)$ is true; that is, suppose any pile of j stones where $1 \leq j \leq k-1$, the sum of all products of the sum and product of each pair of split piles is $(j^3 - j)/3$. Show that this implies the sum of the products of the sums and products is $(k^3 - k)/3$.

First, divide the pile of k stones into two piles of j and $k - j$ stones. Then, the sum of all the products of sums and products equals the sum of the product of $j + (k - j)$ and $j \cdot (k - j)$ along with all the remaining sums. However, since both j and $k - j$ are between 1 and $k - 1$, the induction hypothesis applies and the sum of the products of sums and products is

$$\begin{aligned}
&= (j + (k-j))j(k-j) + \frac{j^3 - j}{3} + \frac{(k-j)^3 - (k-j)}{3} \\
&= \frac{3(jk^2 - j^2k) + j^3 - j + (k-j)^3 - (k-j)}{3} \\
&= \frac{3jk^2 - 3j^2k + j^3 - j + k^3 - 3jk^2 + 3j^2k - j^3 - k + j}{3} \\
&= \frac{k^3 - k}{3}.
\end{aligned}$$

Thus, since both the **Basis Step** and the **Inductive Step** have been shown to be true, $(\forall n \geq 1)\ P(n)$.

Additional Questions

Consider the following two variations on splitting a pile of n objects. In each case try a few examples, conjecture what the solution should be (again, it turns out to be a function of n) and then use the strong form of the principle of mathematical induction to prove your conjecture. As in the problem statement above, suppose at each splitting the sizes of the two smaller piles are r and s.

1. Split the pile of n objects according to the rules above. At each splitting compute the sum of the reciprocals of the two smaller piles: $(\frac{1}{r} + \frac{1}{s})$. Once there are n piles, multiply all of the sums computed. Again, the result will always be the same no matter how each of the piles is split into two smaller piles.

2. Split the pile of n objects according to the rules above. At each splitting compute the following combinatorial number: $\binom{r+s}{r}$. Once there are n piles, multiply all the combinatorial numbers computed.

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