


Richard Courant - Fritz John

Introduction to Calculus and Analysis I

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Richard Courant • Fritz John

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Volume I

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Richard Courant • Fritz John

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Richard Courant Fritz John

Introduction to Calculus and Analysis

Volume I

With 204 Illustrations



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Preface

During the latter part of the seventeenth century the new mathematical analysis emerged as the dominating force in mathematics. It is characterized by the amazingly successful operation with infinite processes or limits. Two of these processes, differentiation and integration, became the core of the systematic Differential and Integral Calculus, often simply called "Calculus," basic for all of analysis.

The importance of the new discoveries and methods was immediately felt and caused profound intellectual excitement. Yet, to gain mastery of the powerful art appeared at first a formidable task, for the available publications were scanty, unsystematic, and often lacking in clarity. Thus, it was fortunate indeed for mathematics and science in general that leaders in the new movement soon recognized the vital need for writing textbooks aimed at making the subject accessible to a public much larger than the very small intellectual elite of the early days. One of the greatest mathematicians of modern times, Leonard Euler, established in introductory books a firm tradition and those books of the eighteenth century have remained sources of inspiration until today, even though much progress has been made in the clarification and simplification of the material.

After Euler, one author after the other adhered to the separation of differential calculus from integral calculus, thereby obscuring a key point, the reciprocity between differentiation and integration. Only in 1927 when the first edition of R. Courant's German *Vorlesungen über Differential und Integralrechnung*, appeared in the Springer-Verlag was this separation eliminated and the calculus presented as a unified subject.

From that German book and its subsequent editions the present work originated. With the cooperation of James and Virginia McShane a greatly expanded and modified English edition of the "Calculus" was prepared and published by Blackie and Sons in Glasgow since 1934, and

distributed in the United States in numerous reprintings by Interscience-Wiley.

During the years it became apparent that the need of college and university instruction in the United States made a rewriting of this work desirable. Yet, it seemed unwise to tamper with the original versions which have remained and still are viable.

Instead of trying to remodel the existing work it seemed preferable to supplement it by an essentially new book in many ways related to the European originals but more specifically directed at the needs of the present and future students in the United States. Such a plan became feasible when Fritz John, who had already greatly helped in the preparation of the first English edition, agreed to write the new book together with R. Courant.

While it differs markedly in form and content from the original, it is animated by the same intention: To lead the student directly to the heart of the subject and to prepare him for active application of his knowledge. It avoids the dogmatic style which conceals the motivation and the roots of the calculus in intuitive reality. To exhibit the interaction between mathematical analysis and its various applications and to emphasize the role of intuition remains an important aim of this new book. Somewhat strengthened precision does not, as we hope, interfere with this aim.

Mathematics presented as a closed, linearly ordered, system of truths without reference to origin and purpose has its charm and satisfies a philosophical need. But the attitude of introverted science is unsuitable for students who seek intellectual independence rather than indoctrination; disregard for applications and intuition leads to isolation and atrophy of mathematics. It seems extremely important that students and instructors should be protected from smug purism.

The book is addressed to students on various levels, to mathematicians, scientists, engineers. It does not pretend to make the subject easy by glossing over difficulties, but rather tries to help the genuinely interested reader by throwing light on the interconnections and purposes of the whole.

Instead of obstructing the access to the wealth of facts by lengthy discussions of a fundamental nature we have sometimes postponed such discussions to appendices in the various chapters.

Numerous examples and problems are given at the end of various chapters. Some are challenging, some are even difficult; most of them supplement the material in the text. In an additional pamphlet more

problems and exercises of a routine character will be collected, and moreover, answers or hints for the solutions will be given.

Many colleagues and friends have been helpful. Albert A. Blank not only greatly contributed incisive and constructive criticism, but he also played a major role in ordering, augmenting, and sifting of the problems and exercises, and moreover he assumed the main responsibility for the pamphlet. Alan Solomon helped most unselfishly and effectively in all phases of the preparation of the book. Thanks is also due to Charlotte John, Anneli Tax, R. Richtmyer, and other friends, including James and Virginia McShane.

The first volume is concerned primarily with functions of a *single* variable, whereas the second volume will discuss the more ramified theories of calculus for functions of several variables.

A final remark should be addressed to the student reader. It might prove frustrating to attempt mastery of the subject by studying such a book page by page following an even path. Only by selecting shortcuts first and returning time and again to the same questions and difficulties can one gradually attain a better understanding from a more elevated point.

An attempt was made to assist users of the book by marking with an asterisk some passages which might impede the reader at his first attempt. Also some of the more difficult problems are marked by an asterisk.

We hope that the work in the present new form will be useful to the young generation of scientists. We are aware of many imperfections and we sincerely invite critical comment which might be helpful for later improvements.

Richard Courant
Fritz John

June 1965

Contents

Chapter 1	Introduction	1
1.1	The Continuum of Numbers	1
	a. The System of Natural Numbers and Its Extension. Counting and Measuring, 1	
	b. Real Numbers and Nested Intervals, 7	
	c. Decimals; Fractions. Bases Other Than Ten, 9	
	d. Definition of Neighborhood, 12	
	e. Inequalities, 12	
1.2	The Concept of Function	17
	a. Mapping-Graph, 18	
	b. Definition of the Concept of Functions of a Continuous Variable. Domain and Range of a Function, 21	
	c. Graphical Representation. Monotone Functions, 24	
	d. Continuity, 31	
	e. The Intermediate Value Theorem. Inverse Functions, 44	
1.3	The Elementary Functions	47
	a. Rational Functions, 47	
	b. Algebraic Functions, 49	
	c. Trigonometric Functions, 49	
	d. The Exponential Function and the Logarithm, 51	
	e. Compound Functions, Symbolic Products, Inverse Functions, 52	
1.4	Sequences	55
1.5	Mathematical Induction	57

1.6 The Limit of a Sequence	60
a. $a_n = \frac{1}{n}$, 61	b. $a_{2n} = \frac{1}{2n}$; $a_{2n+1} = \frac{1}{2n+1}$, 62
c. $a_n = \frac{n}{n+1}$, 63	d. $a_n = \sqrt[n]{p}$, 64
e. $a_n = a^n$, 65	
f. Geometrical Illustration of the Limits of a^n and $\sqrt[n]{p}$, 65	g. The Geometric Series, 67
h. $a_n = \sqrt[n]{n}$, 69	i. $a_n = \sqrt[n]{n+1} - \sqrt[n]{n}$, 69
1.7 Further Discussion of the Concept of Limit	70
a. Definition of Convergence and Divergence, 70	
b. Rational Operations with Limits, 71	
c. Infinite Convergence Tests: Monotone Sequences, 73	d. Infinite Series and the Summation Symbol, 75
e. The Number e , 77	f. The Number e as a Limit, 80
1.8 The Concept of Limit for Functions of a Continuous Variable	82
a. Some Remarks about the Elementary Functions, 86	
<i>Supplements</i>	87
S.1 Limits and the Number Concept	89
a. The Rational Numbers, 89	b. Real Numbers Determined by Nested Sequences of Rational Intervals, 90
c. Order, Limits, and Arithmetic Operations for Real Numbers, 92	d. Completeness of the Number Continuum. Compactness of Closed Intervals. Convergence Criteria, 94
e. Least Upper Bound and Greatest Lower Bound, 97	f. Denumerability of the Rational Numbers, 98
S.2 Theorems on Continuous Functions	99
S.3 Polar Coordinates	101
S.4 Remarks on Complex Numbers	103
PROBLEMS	106

Chapter 2	<i>The Fundamental Ideas of the Integral and Differential Calculus</i>	119
2.1	The Integral	120
	a. Introduction, 120 b. The Integral as an Area, 121 c. Analytic Definition of the Integral. Notations, 122	
2.2	Elementary Examples of Integration	128
	a. Integration of Linear Function, 128 b. Integration of x^n , 130 c. Integration of x^n for Integers $n \neq -1$, 131 d. Integration of x^n for Rational n Other Than -1 , 134 e. Integration of $\sin x$ and $\cos x$, 135	
2.3	Fundamental Rules of Integration	136
	a. Additivity, 136 b. Integral of a Sum of a Product with a Constant, 137 c. Estimating Integrals, 138. d. The Mean Value Theorem for Integrals, 139	
2.4	The Integral as a Function of the Upper Limit (Indefinite Integral)	143
2.5	Logarithm Defined by an Integral	145
	a. Definition of the Logarithm Function, 145 b. The Addition Theorem for Logarithms, 147	
2.6	Exponential Function and Powers	149
	a. The Logarithm of the Number e , 149 b. The Inverse Function of the Logarithm. The Exponential Function, 150 c. The Exponential Function as Limit of Powers, 152 d. Definition of Arbitrary Powers of Positive Numbers, 152 e. Logarithms to Any Base, 153	
2.7	The Integral of an Arbitrary Power of x	154
2.8	The Derivative	155
	a. The Derivative and the Tangent, 156 b. The Derivative as a Velocity, 162	

	c. Examples of Differentiation, 163	d. Some Fundamental Rules for Differentiation, 165	
	e. Differentiability and Continuity of Functions, 166	f. Higher Derivatives and Their Significance, 169	
	g. Derivative and Difference Quotient. Leibnitz's Notation, 171	h. The Mean Value Theorem of Differential Calculus, 173	
	i. Proof of the Theorem, 175	j. The Approximation of Functions by Linear Functions. Definition of Differentials, 179	
	k. Remarks on Applications to the Natural Sciences, 183		
2.9	The Integral, the Primitive Function, and the Fundamental Theorems of the Calculus		184
	a. The Derivative of the Integral, 184	b. The Primitive Function and Its Relation to the Integral, 186	
	c. The Use of the Primitive Function for Evaluation of Definite Integrals, 189	d. Examples, 191	
	<i>Supplement: The Existence of the Definite Integral of a Continuous Function</i>		192
	PROBLEMS		196
Chapter 3	The Techniques of Calculus		201
	<i>Part A: Differentiation and Integration of the Elementary Functions</i>		201
3.1	The Simplest Rules for Differentiation and Their Applications		201
	a. Rules for Differentiation, 201		
	b. Differentiation of the Rational Functions, 204		
	c. Differentiation of the Trigonometric Functions, 205		
3.2	The Derivative of the Inverse Function		206
	a. General Formula, 206	b. The Inverse of the n th Power; the n th Root, 210	
	c. The Inverse Trigonometric Functions--		

	Multivaluedness, 210	d. The Corresponding Integral Formulas, 215	e. Derivative and Integral of the Exponential Function, 216
3.3	Differentiation of Composite Functions	217	
	a. Definitions, 217	b. The Chain Rule, 218	
	c. The Generalized Mean Value Theorem of the Differential Calculus, 223		
3.4	Some Applications of the Exponential Function	223	
	a. Definition of the Exponential Function by Means of a Differential Equation, 223		
	b. Interest Compounded Continuously. Radioactive Disintegration, 224		
	c. Cooling or Heating of a Body by a Surrounding Medium, 225		
	d. Variation of the Atmospheric Pressure with the Height above the Surface of the Earth, 226		
	e. Progress of a Chemical Reaction, 227		
	f. Switching an Electric Circuit on or off, 228		
3.5	The Hyperbolic Functions	228	
	a. Analytical Definition, 228	b. Addition Theorems and Formulas for Differentiation, 231	
	c. The Inverse Hyperbolic Functions, 232		
	d. Further Analogies, 234		
3.6	Maxima and Minima	236	
	a. Convexity and Concavity of Curves, 236		
	b. Maxima and Minima. Relative Extrema. Stationary Points, 238		
3.7	The Order of Magnitude of Functions	248	
	a. The Concept of Order of Magnitude. The Simplest Cases, 248		
	b. The Order of Magnitude of the Exponential Function and of the Logarithm, 249		
	c. General Remarks, 251		
	d. The Order of Magnitude of a Function in the Neighborhood of an Arbitrary Point, 252		
	e. The Order of Magnitude (or Smallness) of a Function Tending to Zero, 252		
	f. The " O " and " o " Notation for Orders of Magnitude, 253		

APPENDIX	255
A.1 Some Special Functions	255
a. The Function $y = e^{1/x^2}$, 255	
b. The Function $y = e^{1/x}$, 256	
c. The Function $y = \operatorname{tanh} 1/x$, 257	
d. The Function $y = x \operatorname{tanh} 1/x$, 258	
e. The Function $y = x \sin 1/x, y(0) = 0$, 259	
A.2 Remarks on the Differentiability of Functions	259
<i>Part B Techniques of Integration</i>	261
3.8 Table of Elementary Integrals	263
3.9 The Method of Substitution	263
a. The Substitution Formula. Integral of a Composite Function, 263	
b. A Second Derivation of the Substitution Formula, 268	
c. Examples. Integration Formulas, 270	
3.10 Further Examples of the Substitution Method	271
3.11 Integration by Parts	274
a. General Formula, 274	
b. Further Examples of Integration by Parts, 276	
c. Integral Formula for $(b + \sqrt{a})$, 278	
d. Recursive Formulas, 278	
*e. Wallis's Infinite Product for π , 280	
3.12 Integration of Rational Functions	282
a. The Fundamental Types, 283	
b. Integration of the Fundamental Types, 284	
c. Partial Fractions, 286	
d. Examples of Resolution into Partial Fractions. Method of Undetermined Coefficients, 288	
3.13 Integration of Some Other Classes of Functions	290
a. Preliminary Remarks on the Rational Representation of the Circle and the Hyperbola, 290	
b. Integration of Arcs $x, \sin x$, 293	
c. Integration of	

$R(\cosh x, \sinh x)$, 294	d. Integration of	
$R(x, \sqrt{1-x^2})$, 294	e. Integration of	
$R(x, \sqrt{x^2-1})$, 295	f. Integration of	
$R(x, \sqrt{x^2+1})$, 295	g. Integration of	
$R(x, \sqrt{ax^2+2bx+c})$, 295	h. Further	
Examples of Reduction to Integrals of Rational Functions, 296	i. Remarks on the Examples,	
297		
<i>Part C. Further Steps in the Theory of Integral Calculus</i>		298
3.14 Integrals of Elementary Functions		298
a. Definition of Functions by Integrals, Elliptic Integrals and Functions, 298	b. On	
Differentiation and Integration, 300		
3.15 Extension of the Concept of Integral		301
a. Introduction. Definition of "Improper" Integrals, 301	b. Functions with Infinite Discontinuities, 303	
c. Interpretation as Area, 304	d. Tests for Convergence, 305	
e. Infinite Interval of Integration, 306	f. The Gamma Function, 308	
g. The Dirichlet Integral, 309	h. Substitution. Fresnel Integrals, 310	
3.16 The Differential Equations of the Trigonometric Functions		312
a. Introductory Remarks on Differential Equations, 312	b. Sin x and cos x defined by a Differential Equation and Initial Conditions, 312	
PROBLEMS		314
<i>Chapter 4 Applications in Physics and Geometry</i>		324
4.1 Theory of Plane Curves		324
a. Parametric Representation, 324	b. Change of Parameters, 326	
c. Motion along a Curve. Time as the Parameter. Example of the		

	Cycloid, 328	d. Classifications of Curves, Orientation, 333	
		e. Derivatives, Tangent and Normal, in Parametric Representation, 343	
	f. The Length of a Curve, 349	g. The Arc Length as a Parameter, 352	
		h. Curvature, 354	
		i. Change of Coordinate Axes, Invariance, 360	
		j. Uniform Motion in the Special Theory of Relativity, 363	
		k. Integrals Expressing Area within Closed Curves, 365	
	l. Center of Mass and Moment of a Curve, 373		
	m. Area and Volume of a Surface of Revolution, 374	n. Moment of Inertia, 375	
4.2	Examples		376
	a. The Common Cycloid, 376	b. The Catenary, 378	
		c. The Ellipse and the Lemniscate, 378	
4.3	Vectors in Two Dimensions		379
	a. Definition of Vectors by Translation, Notions, 380	b. Addition and Multiplication of Vectors, 384	
		c. Variable Vectors, Their Derivatives, and Integrals, 392	
		d. Application to Plane Curves: Direction, Speed, and Acceleration, 394	
4.4	Motion of a Particle under Given Forces		397
	a. Newton's Law of Motion, 397	b. Motion of Falling Bodies, 398	
		c. Motion of a Particle Constrained to a Given Curve, 400	
4.5	Free Fall of a Body Resisted by Air		402
4.6	The Simplest Type of Elastic Vibration		404
4.7	Motion on a Given Curve		405
	a. The Differential Equation and Its Solution, 405	b. Particle Sliding down a Curve, 407	
		c. Discussion of the Motion, 409	
		d. The Ordinary Pendulum, 410	
		e. The Cycloidal Pendulum, 411	

4.8 Motion in a Gravitational Field	413
a. Newton's Universal Law of Gravitation, 413	
b. Circular Motion about the Center of Attraction, 415	
c. Radial Motion—Escape Velocity, 416	
4.9 Work and Energy	418
a. Work Done by Forces during a Motion, 418	
b. Work and Kinetic Energy. Conservation of Energy, 420	
c. The Mutual Attraction of Two Masses, 421	
d. The Stretching of a Spring, 423	
e. The Charging of a Condenser, 423	
APPENDIX	424
A.1 Properties of the Riemann Integral	424
A.2 Areas Bounded by Closed Curves, Indices	430
PROBLEMS	435
<i>Chapter 5</i> Taylor's Expansion	440
5.1 Introduction: Power Series	440
5.2 Expansion of the Logarithm and the Inverse Tangent	442
a. The Logarithm, 442	
b. The Inverse Tangent, 444	
5.3 Taylor's Theorem	446
a. Taylor's Representation of Polynomials, 446	
b. Taylor's Formula for Nonpolynomial Functions, 446	
5.4 Expression and Estimates for the Remainder	447
a. Cauchy's and Lagrange's Expressions, 447	
b. An Alternative Derivation of Taylor's Formula, 450	
5.5 Expansions of the Elementary Functions	453
a. The Exponential Function, 453	

b. Expansion of $\sin x$, $\cos x$, $\sinh x$, $\cosh x$, 454	
c. The Binomial Series, 456	
5.6 Geometrical Applications 457	
a. Contact of Curves, 458	
b. On the Theory of Relative Maxima and Minima, 461	
APPENDIX I 462	
A.I.1 Example of a Function Which Cannot Be Expanded in a Taylor Series 462	
A.I.2 Zeros and Infinites of Functions 463	
a. Zeros of Order n , 463	
b. Infinity of Order v , 463	
A.I.3 Indeterminate Expressions 464	
A.I.4 The Convergence of the Taylor Series of a Function with Nonnegative Derivatives of all Orders 467	
APPENDIX II INTERPOLATION 470	
A.II.1 The Problem of Interpolation. Uniqueness 470	
A.II.2 Construction of the Solution. Newton's Interpolation Formula 471	
A.II.3 The Estimate of the Remainder 474	
A.II.4 The Lagrange Interpolation Formula 476	
PROBLEMS 477	
 Chapter 6 Numerical Methods 481	
6.1 Computation of Integrals 482	
a. Approximation by Rectangles, 482	
b. Refined Approximations—Simpson's Rule, 483	

6.2 Other Examples of Numerical Methods	490
a. The "Calculus of Errors", 490	
b. Calculation of π , 492	
c. Calculation of Logarithms, 493	
6.3 Numerical Solution of Equations	494
a. Newton's Method, 495	
b. The Rule of False Position, 497	
c. The Method of Iteration, 499	
d. Iterations and Newton's Procedure, 502	
APPENDIX	504
A.1 Stirling's Formula	504
PROBLEMS	507
Chapter 7 <i>Infinite Sums and Products</i>	510
7.1 The Concepts of Convergence and Divergence	511
a. Basic Concepts, 511	
b. Absolute Convergence and Conditional Convergence, 513	
c. Rearrangement of Terms, 517	
d. Operations with Infinite Series, 520	
7.2 Tests for Absolute Convergence and Divergence	520
a. The Comparison Test. Majorants, 520	
b. Convergence Tested by Comparison with the Geometric Series, 521	
c. Comparison with an Integral, 524	
7.3 Sequences of Functions	526
a. Limiting Processes with Functions and Curves, 527	
7.4 Uniform and Nonuniform Convergence	529
a. General Remarks and Definitions, 529	
b. A Test for Uniform Convergence, 534	
c. Continuity of the Sum of a Uniformly Convergent Series of Continuous Functions, 536	
d. Integration of Uniformly Convergent Series, 536	
e. Differentiation of Infinite Series, 538	

7.5 Power Series	540
a. Convergence Properties of Power Series Interval of Convergence, 540	
b. Integration and Differentiation of Power Series, 542	
c. Operations with Power Series, 543	
d. Uniqueness of Expansion, 544	
e. Analytic Functions, 545	
7.6 Expansion of Given Functions in Power Series. Method of Undetermined Coefficients. Examples	546
a. The Exponential Function, 546	
b. The Binomial Series, 546	
c. The Series for arc sin x , 549	
d. The Series for ar sinh $x = \log [x + \sqrt{1 + x^2}]$, 549	
e. Example of Multiplication of Series, 550	
f. Example of Term-by-Term Integration (Elliptic Integral), 550	
7.7 Power Series with Complex Terms	551
a. Introduction of Complex Terms into Power Series. Complex Representations of the Trigonometric Function, 551	
b. A Glance at the General Theory of Functions of a Complex Variable, 553	
APPENDIX	555
A.1 Multiplication and Division of Series	555
a. Multiplication of Absolutely Convergent Series, 555	
b. Multiplication and Division of Power Series, 556	
A.2 Infinite Series and Improper Integrals	557
A.3 Infinite Products	559
A.4 Series Involving Bernoulli Numbers	562
PROBLEMS	564

Chapter 8	Trigonometric Series	571
8.1	Periodic Functions	572
	a. General Remarks. Periodic Extension of a Function, 572	
	b. Integrals Over a Period, 573	
	c. Harmonic Vibrations, 574	
8.2	Superposition of Harmonic Vibrations	576
	a. Harmonics. Trigonometric Polynomials, 576	
	b. Beats, 577	
8.3	Complex Notation	582
	a. General Remarks, 582	
	b. Application to Alternating Currents, 583	
	c. Complex Notation for Trigonometrical Polynomials, 585	
	d. A Trigonometric Formula, 586	
8.4	Fourier Series	587
	a. Fourier Coefficients, 587	
	b. Basic Lemma, 588	
	c. Proof of $\int_0^{\pi} \frac{\sin^2 z}{z} dz = \frac{\pi}{2}$, 589	
	d. Fourier Expansion for the Function $\varphi(x) = x$, 591	
	e. The Main Theorem on Fourier Expansion, 593	
8.5	Examples of Fourier Series	598
	a. Preliminary Remarks, 598	
	b. Expansion of the Function $\varphi(x) = x^2$, 598	
	c. Expansion of $x \cos x$, 598	
	d. The Function $f(x) = x $, 600	
	e. A Piecewise Constant Function, 600	
	f. The Function $\sin x $, 601	
	g. Expansion of $\cos \nu x$. Resolution of the Cotangent into Partial Fractions. The Infinite Product for the Sine, 602	
	h. Further examples, 603	
8.6	Further Discussion of Convergence	604
	a. Results, 604	
	b. Bessel's Inequality, 604	

e. Proof of Corollaries (a), (b), and (c).	605
d. Order of Magnitude of the Fourier Coefficients: Differentiation of Fourier Series.	607
8.7 Approximation by Trigonometric and Rational Polynomials	608
a. General Remark on Representations of Functions, 608	
b. Weierstrass Approximation Theorem, 608	
c. Fejers Trigonometric Approximation of Fourier Polynomials by Arithmetical Means, 610	
d. Approximation in the Mean and Parseval's Relation, 612	
APPENDIX I	614
A.I.1 Stretching of the Period Interval. Fourier's Integral Theorem	614
A.I.2 Gibbs's Phenomenon at Points of Discontinuity	616
A.I.3 Integration of Fourier Series	618
APPENDIX II	619
A.II.1 Bernoulli Polynomials and Their Applications	619
a. Definition and Fourier Expansion, 619	
b. Generating Functions and the Taylor Series of the Trigonometric and Hyperbolic Cotangent, 621	
c. The Euler-Maclaurin Summation Formula, 624	
d. Applications, Asymptotic Expressions, 626	
e. Sums of Power: Recursion Formula for Bernoulli Numbers, 628	
f. Euler's Constant and Stirling's Series, 629	
PROBLEMS	631

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