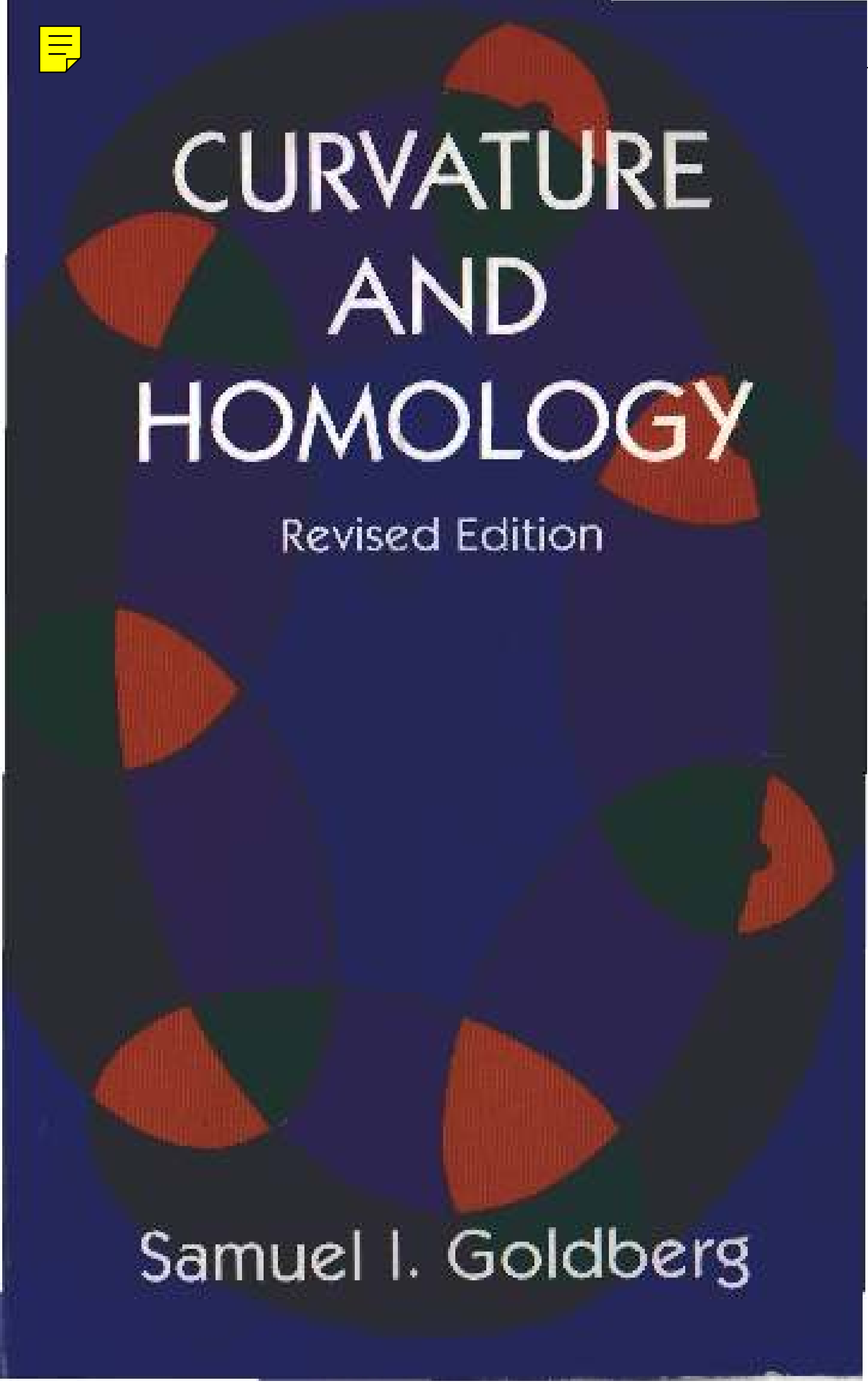
CURVATURE
AND
HOMOLOGY
Revised Edition
Samuel I. Goldberg

CURVATURE AND HOMOLOGY

Revised Edition

Samuel I. Goldberg

Mathematicians interested in the curvature properties of Riemannian manifolds and their homologic structures, an increasingly important and specialized branch of differential geometry, will welcome this excellent teaching text. Revised and expanded by its well-known author, this volume offers a systematic and self-contained treatment of subjects such as the topology of differentiable manifolds, curvature and homology of Riemannian manifolds, compact Lie groups, complex manifolds, and the curvature and homology of Kaehler manifolds.

In addition to a new preface, this edition includes five new appendices concerning holomorphic bisectional curvature, the Gauss-Bonnet theorem, some applications of the generalized Gauss-Bonnet theorem, an application of Bochner's lemma, and the Kodaira vanishing theorems. Geared toward readers familiar with standard courses in linear algebra, real and complex variables, differential equations, and point-set topology, the book features helpful exercises at the end of each chapter that supplement and clarify the text.

This lucid and thorough treatment—hailed by *Nature* magazine as "... a valuable survey of recent work and of probable lines of future progress"—includes material unavailable elsewhere and provides an excellent resource for both students and teachers.

Unabridged Dover (1998) corrected republication of the work originally published by Academic Press, New York, 1962. New Preface. Introduction. Five new appendixes. Bibliography. Indices. 416pp. 5⅜ x 8½. Paperbound.

Free Dover Mathematics and Science Catalog (59065-8) available upon request.

ISBN 0-486-40207-X

$14.95 IN USA

Curvature

and

Homology

SAMUEL I. GOLDBERG

Department of Mathematics
University of Illinois, Urbana-Champaign

DOVER PUBLICATIONS, INC.
Mineola, New York

To my parents and my wife

Copyright

Published in Canada by General Publishing Company, Ltd., 30 Lesmill Road, Don Mills, Toronto, Ontario.
Published in the United Kingdom by Constable and Company, Ltd., 3 The Lanchesters, 162–164 Fulham Palace Road, London W6 9ER.

Bibliographical Note

This Dover enlarged edition, first published in 1998, is an unabridged, corrected, and enlarged republication of the second printing (1970) of the work first published in 1962 by Academic Press, N.Y., as Volume 11 in the series *Pure and Applied Mathematics*. Five new Appendices, a new Preface, and additional reference titles have been added to this edition.

Library of Congress Cataloging-in-Publication Data

Goldberg, Samuel I.
Curvature and homology / Samuel I. Goldberg. — Enl. ed.
p. cm.
Includes bibliographical references and indexes.
ISBN 0-486-40207-X (pbk.)
1. Curvature. 2. Homology theory. 3. Geometry, Riemannian. I. Title.
QA645.G6 1998
516.3'62—dc21 98-22211
CIP

Manufactured in the United States of America
Dover Publications, Inc., 31 East 2nd Street, Mineola, N.Y. 11501

PREFACE TO THE ENLARGED EDITION

Originally, in the first edition of this work, it was the author's purpose to provide a self-contained treatment of Curvature and Homology. Subsequently, it became apparent that the more important applications are to Kaehler manifolds, particularly the Kodaira vanishing theorems, which appear in Chapter VI. To make this chapter comprehensible, Appendices F and I have been added to this new edition. In these Appendices, the Chern classes are defined and the Euler characteristic is given by the Gauss-Bonnet formula — the latter being applied in Appendix G. Several important recent developments are presented in Appendices E and H. In Appendix E, the differential geometric technique due to Bochner gives rise to an important result that was established by Siu and Yau in 1980. The same method is applied in Appendix H to F-structures over negatively curved spaces.

S. I. Goldberg

Urbana, Illinois
February, 1998

PREFACE

The purpose of this book is to give a systematic and "self-contained" account along modern lines of the subject with which the title deals, as well as to discuss problems of current interest in the field. With this statement the author wishes to recall another book, "Curvature and Betti Numbers," by K. Yano and S. Bochner; this tract is aimed at those already familiar with differential geometry, and has served admirably as a useful reference during the nine years since its appearance. In the present volume, a coordinate free treatment is presented wherever it is considered feasible and desirable. On the other hand, the index notation for tensors is employed whenever it seems to be more adequate.

The book is intended for the reader who has taken the standard courses in linear algebra, real and complex variables, differential equations, and point-set topology. Should he lack an elementary knowledge of algebraic topology, he may accept the results of Chapter II and proceed from there. In Appendix C he will find that some knowledge of Hilbert space methods is required. This book is also intended for the more seasoned mathematician, who seeks familiarity with the developments in this branch of differential geometry in the large. For him to feel at home a knowledge of the elements of Riemannian geometry, Lie groups, and algebraic topology is desirable.

The exercises are intended, for the most part, to supplement and to clarify the material wherever necessary. This has the advantage of maintaining emphasis on the subject under consideration. Several might well have been explained in the main body of the text, but were omitted in order to focus attention on the main ideas. The exercises are also devoted to miscellaneous results on the homology properties of rather special spaces, in particular, δ-pinched manifolds, locally convex hypersurfaces, and minimal varieties. The inexperienced reader should not be discouraged if the exercises appear difficult. Rather, should he be interested, he is referred to the literature for clarification.

References are enclosed in square brackets. Proper credit is almost always given except where a reference to a later article is either more informative or otherwise appropriate. Cross references appear as (6.8.2) referring to Chapter VI, Section 8, Formula 2 and also as (VI.A.3) referring to Chapter VI, Exercise A, Problem 3.

The author owes thanks to several colleagues who read various parts of the manuscript. He is particularly indebted to Professor M. Obata, whose advice and diligent care has led to many improvements. Professor R. Bishop suggested some exercises and further additions. Gratitude is

also extended to Professors R. G. Bartle and A. Heller for their critical reading of Appendices A and C as well as to Dr. L. McCulloh and Mr. R. Vogt for assisting with the proofs. For the privilege of attending his lectures on Harmonic Integrals at Harvard University, which led to the inclusion of Appendix A, thanks are extended to Professor L. Ahlfors. Finally, the author expresses his appreciation to Harvard University for the opportunity of conducting a seminar on this subject.

It is a pleasure to acknowledge the invaluable assistance received in the form of partial financial support from the Air Force Office of Scientific Research.

S. I. Goldberg

Urbana, Illinois
February, 1962

CONTENTS

Appendix F

THE GAUSS-BONNET THEOREM

Appendix G

SOME APPLICATIONS OF THE GENERALIZED GAUSS-BONNET THEOREM

Appendix H

AN APPLICATION OF BOCHNER'S LEMMA

Appendix I

THE KODAIRA VANISHING THEOREMS

NOTATION INDEX

The symbols used have gained general acceptance with some exceptions. In particular, R and C are the fields of real and complex numbers, respectively. (In § 7.1, the same letter C is employed as an operator and should cause no confusion.) The commonly used symbols $\in, \cup, \cap, \cong$, sup, inf, are not listed. The exterior or Grassman algebra of a vector space V (over R or C) is written as $\Lambda(V)$. By $\Lambda^p(V)$ is meant the vector space of its elements of degree p and $\wedge$ denotes multiplication in $\Lambda(V)$. The elements of $\Lambda(V)$ are designated by Greek letters. The symbol M is reserved for a topological manifold, T_P its tangent space at a point $P \in M$ (in case M is a differentiable manifold) and T_P^* the dual space (of covectors). The space of tangent vector fields is denoted by T and its dual by T^*. The Lie bracket of tangent vectors X and Y is written as $[X, Y]$. Tensors are generally denoted by Latin letters. For example, the metric tensor of a Riemannian manifold will usually be denoted by g. The covariant form of X (with respect to g) is designated by the corresponding Greek symbol ξ. The notation for composition of functions (maps) employed is flexible. It is sometimes written as $g \cdot f$ and at other times the dot is not present. The dot is also used to denote the (local) scalar product of vectors (relative to g). However, no confusion should arise

Symbol		*Page*
$D(M)$, $D^p(M)$:	d-cohomology ring, p-dimensional d-cohomology group	15
C_p, Z_p, B_p, H_p :		57, 58
S_p, S_p^u, S_p^0, SH_p :		61
C^p, Z^p, B^p, H^p :		59, 60
b_p :	p^{th} betti number	60, 63
$b_{q,r}$:	complex dimension of $\wedge_H^{q,r}$	177
d :	differential operator	14, 168
d', d'' :	differential operator of type (1,0), (0,1)	168, 233
∂ :	boundary operator	21, 58, 61
$*$, $*^{-1}$:	star operator, inverse of star operator	66, 70, 97
δ :	co-differential operator	72, 170, 233
δ', δ'' :	co-differential operator of type $(-1,0)$, $(0,-1)$	170
Δ :	Laplace-Beltrami operator	73, 233
H :	Harmonic projector	80, 178
G :	Green's operator	80, 178
φ_*, φ^* :	induced maps of φ	18
$\wedge_c^p$, $\wedge_2^p - \wedge_k^p$:		15, 74
$\wedge_H^p$, $\wedge_K^p$:		74
D_j :	covariant differential operator	24
D_X :	covariant differential operator	192
Q :	Ricci operator	87
$\epsilon(\xi)$:	exterior product by ξ operator	96
$i(X)$:	interior product by X operator	97, 171
$\theta(X)$:	Lie derivative operator	101, 134
J :	almost complex structure tensor	151
$V^{1,0}$, $V^{0,1}$:	space of vectors of bidegree (1,0), (0,1)	152
$\wedge^{q,r}$:	space of exterior forms of bidegree (q, r)	152
$\wedge_H^{q,r}$:	space of harmonic forms of bidegree (q, r)	177
Ω :	fundamental 2-form	165
$L = \epsilon(\Omega)$:		170
$\Lambda = (-1)^p *L*$:		176

$O(n) = O(n, R)$: The subgroup of $Gl(n, R)$ consisting of those matrices a for which ${}^t a = a^{-1}$ where a^{-1} is the inverse of a and ${}^t a$ denotes its transpose : ${}^t(a^i_j) = (a^j_i)$.
$U(n) = \{a \in Gl(n, C) \mid \bar{a} = {}^t a^{-1}\}$, where $\bar{a} = (\overline{a^i_j})$.
$SU(n) = \{a \in U(n) \mid \det(a) = 1\}$.

INTRODUCTION

The most important aspect of differential geometry is perhaps that which deals with the relationship between the curvature properties of a Riemannian manifold and its topological structure. One of the beautiful results in this connection is the generalized Gauss-Bonnet theorem which for orientable surfaces has long been known. In recent years there has been a considerable increase in activity in global differential geometry thanks to the celebrated work of W. V. D. Hodge and the applications of it made by S. Bochner, A. Lichnerowicz, and K. Yano. In the decade since the appearance of Bochner's first papers in this field many fruitful investigations on the subject matter of "curvature and betti numbers" have been inaugurated. The applications are, to some extent, based on a theorem in differential equations due to E. Hopf. The Laplace-Beltrami operator Δ is elliptic and when applied to a function f of class 2 defined on a compact Riemannian manifold M yields the Bochner lemma: "If $\Delta f \geq 0$ everywhere on M, then f is a constant and Δf vanishes identically." Many diverse applications to the relationship between the curvature properties of a Riemannian manifold and its homology structure have been made as a consequence of this "observation." Of equal importance, however, a "dual" set of results on groups of motions is realized.

The existence of harmonic tensor fields over compact orientable Riemannian manifolds depends largely on the signature of a certain quadratic form. The operator Δ introduces curvature, and these properties of the manifold determine to some extent the global structure via Hodge's theorem relating harmonic forms with betti numbers. In Chapter II, therefore, the theory of harmonic integrals is developed to the extent necessary for our purposes. A proof of the existence theorem of Hodge is given (modulo the fundamental differentiability lemma C.1 of Appendix C), and the essential material and information necessary for the treatment and presentation of the subject of curvature and homology is presented. The idea of the proof of the existence theorem is to show that Δ^{-1}—the inverse of the closure of Δ—is a completely continuous operator. The reader is referred to de Rham's book "Variétés Différentiables" for an excellent exposition of this result.

The spaces studied in this book are important in various branches of mathematics. Locally they are those of classical Riemannian geometry, and from a global standpoint they are compact orientable manifolds. Chapter I is concerned with the local structure, that is, the geometry of the space over which the harmonic forms are defined. The properties necessary for an understanding of later chapters are those relating to the differential geometry of the space, and those which are topological properties. The topology of a differentiable manifold is therefore discussed in Chapter II. Since these subjects have been given essentially complete and detailed treatments elsewhere, and since a thorough discussion given here would reduce the emphasis intended, only a brief survey of the bare essentials is outlined. Families of Riemannian manifolds are described in Chapter III, each including the n-sphere and retaining its betti numbers. In particular, a 4-dimensional δ-pinched manifold is a homology sphere provided $\delta > \frac{1}{4}$. More generally, the second betti number of a δ-pinched even-dimensional manifold is zero if $\delta > \frac{1}{4}$.

The theory of harmonic integrals has its origin in an attempt to generalize the well-known existence theorem of Riemann to everywhere finite integrals over a Riemann surface. As it turns out in the generalization a $2n$-dimensional Riemannian manifold plays the part of the Riemann surface in the classical 2-dimensional case although a Riemannian manifold of 2 dimensions is not the same as a Riemann surface. The essential difference lies in the geometry which in the latter case is conformal. In higher dimensions, the concept of a complex analytic manifold is the natural generalization of that of a Riemann surface in the abstract sense. In this generalization concepts such as holomorphic function have an invariant meaning with respect to the given complex structure. Algebraic varieties in a complex projective space P_n have a natural complex structure and are therefore complex manifolds provided there are no "singularities." There exist, on the other hand, examples of complex manifolds which cannot be imbedded in a P_n. A complex manifold is therefore more general than a projective variety. This approach is in keeping with the modern developments due principally to A. Weil.

It is well-known that all orientable surfaces admit complex structures. However, for higher even-dimensional orientable manifolds this is not the case. It is not possible, for example, to define a complex structure on the 4-dimensional sphere. (In fact, it was recently shown that not every topological manifold possesses a differentiable structure.) For a given complex manifold M not much is known about the complex structure itself; all consequences are derived from assumptions which are weaker—the "almost-complex" structure, or stronger—the existence of a "Kaehler metric." The former is an assumption concerning the tangent bundle of M and therefore suitable for fibre space methods, whereas the latter is an assumption on the Riemannian geometry of M, which can be investigated by the theory of harmonic forms. The material of Chapter V is partially concerned with a development of hermitian geometry, in particular, Kaehler geometry along the lines proposed by S. Chern. Its influence on the homology structure of the manifold is discussed in Chapters V and VI. Whereas the homology properties described in Chapter III are similar to those of the ordinary sphere (insofar as betti numbers are concerned), the corresponding properties in Chapter VI are possessed by P_n itself. Families of hermitian manifolds are described, each including P_n and retaining its betti numbers. One of the most important applications of the effect of curvature on homology is to be found in the vanishing theorems due to K. Kodaira. They are essential in the applications of sheaf theory to complex manifolds.

A conformal transformation of a compact Riemann surface is a holomorphic homeomorphism. For compact Kaehler manifolds of higher dimension, an element of the connected component of the identity of the group of conformal transformations is an isometry, and consequently a holomorphic homeomorphism. More generally, an infinitesimal conformal map of a compact Riemannian manifold admitting a harmonic form of constant length is an infinitesimal isometry. Thus, if a compact homogeneous Riemannian manifold admits an infinitesimal non-isometric conformal transformation, it is a homology sphere. Indeed, it is then isometric with a sphere. The conformal transformation group is studied in Chapter III, and in Chapter VII groups of holomorphic as well as conformal homeomorphisms of Kaehler manifolds are investigated.

In Appendix A, a proof of de Rham's theorems based on the concept of a sheaf is given although this notion is not defined. Indeed, the proof is but an adaptation from the general theory of sheaves and a knowledge of the subject is not required.

CHAPTER I

RIEMANNIAN MANIFOLDS

In seeking to generalize the well-known theorem of Riemann on the existence of holomorphic integrals over a Riemann surface, W. V. D. Hodge [30] considers an n-dimensional Riemannian manifold as the space over which a certain class of integrals is defined. Now, a Riemannian manifold of two dimensions is not a Riemann surface, for the geometry of the former is Riemannian geometry whereas that of a Riemann surface is conformal geometry. However, in a certain sense a 2-dimensional Riemannian manifold may be thought of as a Riemann surface. Moreover, conformally homeomorphic Riemannian manifolds of two dimensions define equivalent Riemann surfaces. Conversely, a Riemann surface determines an infinite set of conformally homeomorphic 2-dimensional Riemannian manifolds. Since the underlying structure of a Riemannian manifold is a differentiable structure, we discuss in this chapter the concept of a differentiable manifold, and then construct over the manifold the integrals, tensor fields and differential forms which are basically the objects of study in the remainder of this book.

1.1. Differentiable manifolds

The differential calculus is the main tool used in the study of the geometrical properties of curves and surfaces in ordinary Euclidean space E^3. The concept of a curve or surface is not a simple one, so that in many treatises on differential geometry a rigorous definition is lacking. The discussions on surfaces are further complicated since one is interested in those properties which remain invariant under the group of motions in E^3. This group is itself a 6-dimensional manifold. The purpose of this section is to develop the fundamental concepts of differentiable manifolds necessary for a rigorous treatment of differential geometry.

Given a topological space, one can decide whether a given function defined over it is continuous or not. A discussion of the properties of the classical surfaces in differential geometry requires more than continuity, however, for the functions considered. By a *regular closed surface* S in E^3 is meant an ordered pair $\{S_0, X\}$ consisting of a topological space S_0 and a differentiable map X of S_0 into E^3. As a topological space, S_0 is to be a separable, Hausdorff space with the further properties:

(i) S_0 is compact (that is $X(S_0)$ is closed and bounded);

(ii) S_0 is connected (a topological space is said to be *connected* if it cannot be expressed as the union of two non-empty disjoint open subsets);

(iii) Each point of S_0 has an open neighborhood homeomorphic with E^2. The map $X: P \to (x(P), y(P), z(P))$, $P \in S_0$ where $x(P)$, $y(P)$ and $z(P)$ are differentiable functions is to have rank 2 at each point $P \in S_0$, that is the matrix

$$\begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \end{pmatrix}$$

of partial derivatives must be of rank 2 where u, v are local parameters at P. Let U and V be any two open neighborhoods of P homeomorphic with E^2 and with non-empty intersection. Then, their local parameters or coordinates (cf. definition given below of a differentiable structure) must be related by differentiable functions with non-vanishing Jacobian. It follows that the rank of X is invariant with respect to a change of coordinates.

That a certain amount of differentiability is necessary is clear from several points of view. In the first place, the condition on the rank of X implies the existence of a tangent plane at each point of the surface. Moreover, only those local parameters are "allowed" which are related by differentiable functions.

A regular closed surface is but a special case of a more general concept which we proceed to define.

Roughly speaking, a differentiable manifold is a topological space in which the concept of derivative has a meaning. Locally, the space is to behave like Euclidean space. But first, a topological space M is said to be *separable* if it contains a countable basis for its topology. It is called a *Hausdorff space* if to any two points of M there are disjoint open sets each containing exactly one of the points.

A separable Hausdorff space M of dimension n is said to have a *differentiable structure* of class $k > 0$ if it has the following properties:

(i) Each point of M has an open neighborhood homeomorphic with an open subset in E^n the (number) space of n real variables, that is, there is a finite or countable open covering $\{U_\alpha\}$ and, for each α a homeomorphism $u_\alpha : U_\alpha \to R^n$ of U_α onto an open subset in R^n;

(ii) For any two open sets U_α and U_β with non empty intersection the map $u_\beta u_\alpha^{-1} : u_\alpha(U_\alpha \cap U_\beta) \to R^n$ is of class k (that is, it possesses continuous derivatives of order k) with non-vanishing Jacobian.

The functions defining u_α are called *local coordinates in* U_α. Clearly, one may also speak of structures of class ∞ (that is, structures of class k for every positive integer k) and *analytic* structures (that is, every map $u_\beta u_\alpha^{-1}$ is expressible as a convergent power series in the n variables). The local coordinates constitute an essential tool in the study of M. However, the geometrical properties should be independent of the choice of local coordinates.

The space M with the property (i) will be called a *topological manifold*. We shall generally assume that the spaces considered are connected although many of the results are independent of this hypothesis.

Examples: 1. The Euclidean space E^n is perhaps the simplest example of a topological manifold with a differentiable structure. The identity map I in R^n together with the *unit covering* (R^n, I) is its natural differentiable structure: $(U_1, u_1) = (R^n, I)$.

2. The $(n-1)$-dimensional sphere in E^n defined by the equation

$$\sum_{i=1}^{n} (x^i)^2 = 1. \tag{1.1.1}$$

It can be covered by $2n$ coordinate neighborhoods defined by $x^i > 0$ and $x^i < 0$ $(i = 1, ..., n)$.

3. The general linear group: Let V be a vector space over R (the real numbers) of dimension n and let $\{e_1, ..., e_n\}$ be a basis of V. The group of all linear automorphisms a of V may be expressed as the group of all non-singular matrices (a_i^j);

$$ae_i = a_i^j e_j, \qquad i, j = 1, ..., n \tag{1.1.2}$$

called the *general linear group* and denoted by $GL(n, R)$. We shall also denote it by $GL(V)$ when dealing with more than one vector space. (The Einstein summation convention where repeated indices implies addition has been employed in formula (1.1.2) and, in the sequel we shall adhere to this notation.) The multiplication law is

$$(ab)_i^j = a_i^k b_k^j .$$

$GL(n, R)$ may be considered as an open set (and hence as an open submanifold (cf. §1.5) of R^{n^2}. With this structure (as an analytic manifold), $GL(n, R)$ is a Lie group (cf. §3.6).

Let f be a real-valued continuous function defined in an open subset S of M. Let P be a point of S and U_α a coordinate neighborhood containing P. Then, in $S \cap U_\alpha$, f can be expressed as a function of the local coordinates $u^1, \ldots, u^n$ in U_α. (If $x^1, \ldots, x^n$ are the n coordinate functions on R^n, then $u^i(P) = x^i(u_\alpha(P))$, $i = 1, \ldots, n$ and we may write $u^i = x^i \circ u_\alpha$). The function f is said to be *differentiable* at P if $f(u^1, \ldots, u^n)$ possesses all first partial derivatives at P. The *partial derivative* of f with respect to u^i at P is defined as

$$\left(\frac{\partial f}{\partial u^i}\right)_P = \left(\frac{\partial (f u_\alpha^{-1})}{\partial x^i}\right)_{u_\alpha(P)}.$$

This property is evidently independent of the choice of U_α. The function f is called *differentiable* in S, if it is differentiable at every point of S. Moreover, f is of the form $g \circ u_\alpha$ if the domain is restricted to $S \cap U_\alpha$ where g is a continuous function in $u_\alpha(S \cap U_\alpha) \subset R^n$. Two differentiable structures are said to be *equivalent* if they give rise to the same family of differentiable functions over open subsets of M. This is an equivalence relation. The family of functions of class k determines the differentiable structures in the equivalence class.

A topological manifold M together with an equivalence class of differentiable structures on M is called a *differentiable manifold*. It has recently been shown that not every topological manifold can be given a differentiable structure [44]. On the other hand, a topological manifold may carry differentiable structures belonging to distinct equivalence classes. Indeed, the 7-dimensional sphere possesses several inequivalent differentiable structures [60].

A differentiable mapping f of an open subset S of R^n into R^n is called *sense-preserving* if the Jacobian of the map is positive in S. If, for any pair of coordinate neighborhoods with non-empty intersection, the mapping $u_\beta u_\alpha^{-1}$ is sense preserving, the differentiable structure is said to be *oriented* and, in this case, the differentiable manifold is called *orientable*. Thus, if $f_{\beta\alpha}(x)$ denotes the Jacobian of the map $u_\beta u_\alpha^{-1}$ at $x^i(u_\alpha(P))$, $i = 1, \ldots, n$, then $f_{\gamma\beta}(x) f_{\beta\alpha}(x) = f_{\gamma\alpha}(x)$, $P \in U_\alpha \cap U_\beta \cap U_\gamma$.

The 2-sphere in R^3 is an orientable manifold whereas the real projective plane (the set of lines through the origin in R^3) is not (cf. I.B. 2).

Let M be a differentiable manifold of class k and S an open subset of M. By restricting the functions (of class k) on M to S, the differentiable structure so obtained on S is called an *induced structure* of class k. In particular, on every open subset of R^n there is an induced structure

1.2. Tensors

To every point P of a regular surface S there is associated the tangent plane at P consisting of the tangent vectors to the curves on S through P. A tangent vector t may be expressed as a linear combination of the tangent vectors X_u and X_v "defining" the tangent plane:

$$t = \xi^1 X_u + \xi^2 X_v, \qquad \xi^i \in R, \qquad i = 1, 2. \tag{1.2.1}$$

At this point, we make a slight change in our notation: We put $u^1 = u$, $u^2 = v$, $X_1 = X_u$ and $X_2 = X_v$, so that (1.2.1) becomes

$$t = \xi^i X_i. \tag{1.2.2}$$

Now, in the coordinates $\bar{u}^1, \bar{u}^2$ where the $\bar{u}^i$ are related to the u^i by means of differentiable functions with non-vanishing Jacobian

$$t = \xi^i \frac{\partial \bar{u}^j}{\partial u^i} \bar{X}_j \tag{1.2.3}$$

where $\bar{X} = X(u^1(\bar{u}^1, \bar{u}^2), u^2(\bar{u}^1, \bar{u}^2))$. If we put

$$\bar{\xi}^j = \frac{\partial \bar{u}^j}{\partial u^i} \xi^i \tag{1.2.4}$$

equation (1.2.3) becomes

$$t = \bar{\xi}^j \bar{X}_j. \tag{1.2.5}$$

In classical differential geometry the vector t is called a contravariant vector, the equations of transformation (1.2.4) determining its character.

Guided by this example we proceed to define the notion of contravariant vector for a differentiable manifold M of dimension n. Consider the triple (P, U_α, ξ^i) consisting of a point $P \in M$, a coordinate neighborhood U_α containing P and a set of n real numbers ξ^i. An equivalence relation is defined if we agree that the triples (P, U_α, ξ^i) and $(\bar{P}, U_\beta, \bar{\xi}^i)$ are *equivalent* if $P = \bar{P}$ and

$$\bar{\xi}^j = \left(\frac{\partial \bar{u}^j}{\partial u^i}\right)_{\varphi_\alpha(P)} \xi^i \tag{1.2.6}$$

where the u^i are the coordinates of $\varphi_\alpha(P)$ and $\bar{u}^i$ those of $\varphi_\beta(P)$, $P \in U_\alpha \cap U_\beta$. An equivalence class of such triples is called a *contravariant vector* at P. When there is no danger of confusion we simply speak of the contravariant vector by choosing a particular set of representatives ξ^i

$(i = 1, \ldots, n)$. That the contravariant vectors form a linear space over R is clear. In analogy with surface theory this linear space is called the *tangent space* at P and will be denoted by T_P. (For a rather sophisticated definition of tangent vector the reader is referred to §3.4.)

Let f be a differentiable function defined in a neighborhood of $P \in U_\alpha \cap U_\beta$. Then,

$$\left(\frac{\partial (f\varphi_\alpha^{-1})}{\partial x^i}\right)_{\varphi_\alpha(P)} = \left(\frac{\partial (f\varphi_\beta^{-1})}{\partial x^j}\right)_{\varphi_\beta(P)} \left(\frac{\partial \bar{u}^j}{\partial u^i}\right)_{\varphi_\alpha(P)}. \tag{1.2.7}$$

Now, applying (1.2.6) we obtain

$$\left(\frac{\partial (f\varphi_\alpha^{-1})}{\partial x^i}\right)_{\varphi_\alpha(P)} \xi^i = \left(\frac{\partial (f\varphi_\beta^{-1})}{\partial x^j}\right)_{\varphi_\beta(P)} \bar{\xi}^j. \tag{1.2.8}$$

The equivalence class of "functions" of which the left hand member of (1.2.8) is a representative is commonly called the *directional derivative* of f along the contravariant vector ξ^i. In particular, if the components $\xi^i (i = 1, \ldots, n)$ all vanish except the k^{th} which is 1, the directional derivative is the partial derivative with respect to u^k and the corresponding contravariant vector is denoted by $\partial/\partial u^k$. Evidently, these vectors for all $k = 1, \ldots, n$ form a base of T_P called the *natural base*. On the other hand, the partial derivatives of f in (1.2.8) are representatives of a vector (which we denote by df) in the dual space T_P^* of T_P. The elements of T_P^* are called *covariant vectors* or, simply, *covectors*. In the sequel, when we speak of a covariant vector at P, we will occasionally employ a set of representatives. Hence, if η_i is a covariant vector and ξ^i a contravariant vector the expression $\eta_i\xi^i$ is a *scalar invariant* or, simply *scalar*, that is

$$\bar{\eta}_i\bar{\xi}^i = \eta_i\xi^i, \tag{1.2.9}$$

and so,

$$\eta_i = \frac{\partial \bar{u}^j}{\partial u^i}\bar{\eta}_j \tag{1.2.10}$$

are the equations of transformation defining a covariant vector. We define the *inner product* of a contravariant vector $v = \xi^i$ and a covariant vector $w^* = \eta_i$ by the formula

$$\langle v, w^* \rangle = \eta_i\xi^i. \tag{1.2.11}$$

That the inner product is bilinear is clear. Now, from (1.2.10) we obtain

$$\bar{\eta}_i\, d\bar{u}^i = \eta_i\, du^i \tag{1.2.12}$$

where the $du^i (i = 1, \ldots, n)$ are the differentials of the functions $u^1, \ldots, u^n$.

sample content of Curvature and Homology: Revised Edition

- *download online The Philosophers Plant: An Intellectual Herbarium*
- click Pandora (Vampire Chronicles: New Tales of the Vampires, Book 1)
- download online The Ultimate Time Machine: A Remote Viewer's Perception of Time & Predictions for the New Millennium
- click The Identity of Man (Great Minds Series)
- **download online The End of White World Supremacy: Four Speeches online**
- **American Fairy Tales book**

- http://sidenoter.com/?ebooks/The-Philosophers-Plant--An-Intellectual-Herbarium.pdf
- http://sidenoter.com/?ebooks/Pandora--Vampire-Chronicles--New-Tales-of-the-Vampires--Book-1-.pdf
- http://fitnessfatale.com/freebooks/Truth--The-Broken-Shore--Book-2-.pdf
- http://yachtwebsitedemo.com/books/Complete-Guide-to-Gay---Lesbian-Weddings.pdf
- http://tuscalaural.com/library/A-Mencken-Chrestomathy--His-Own-Selection-of-His-Choicest-Writing.pdf
- http://aneventshop.com/ebooks/Genus.pdf